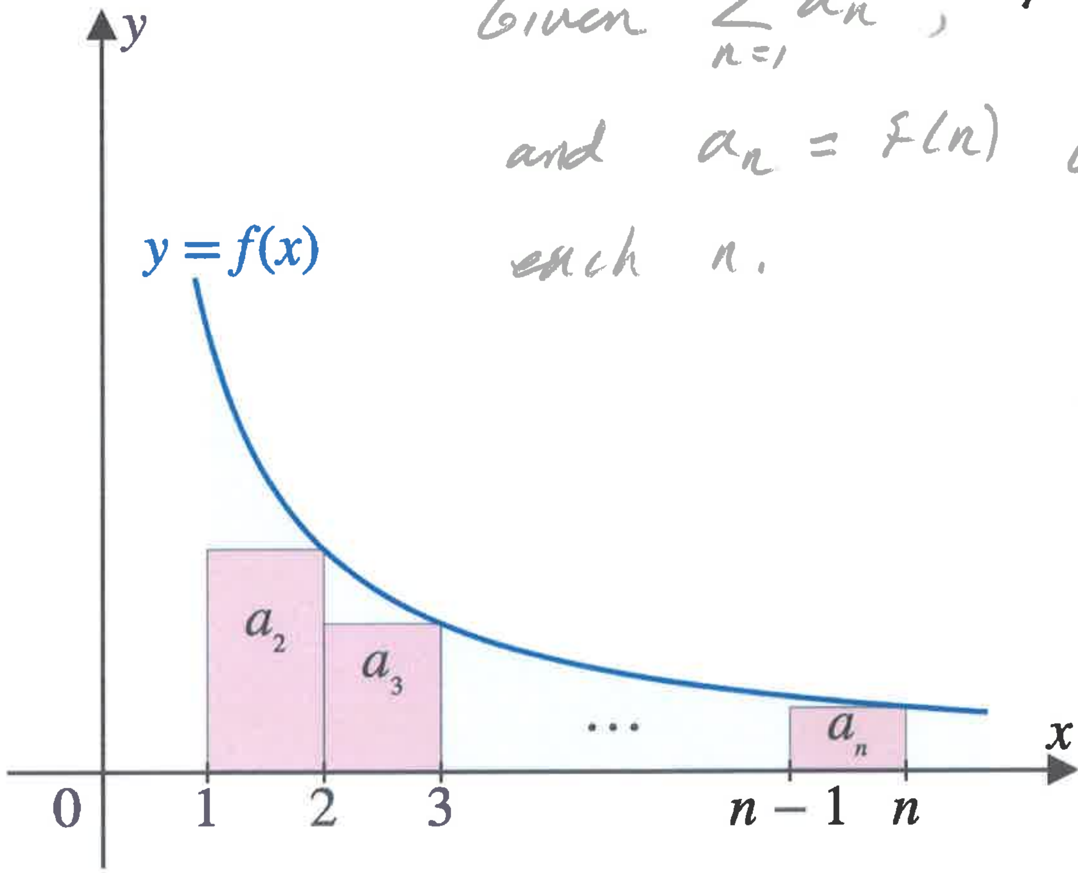
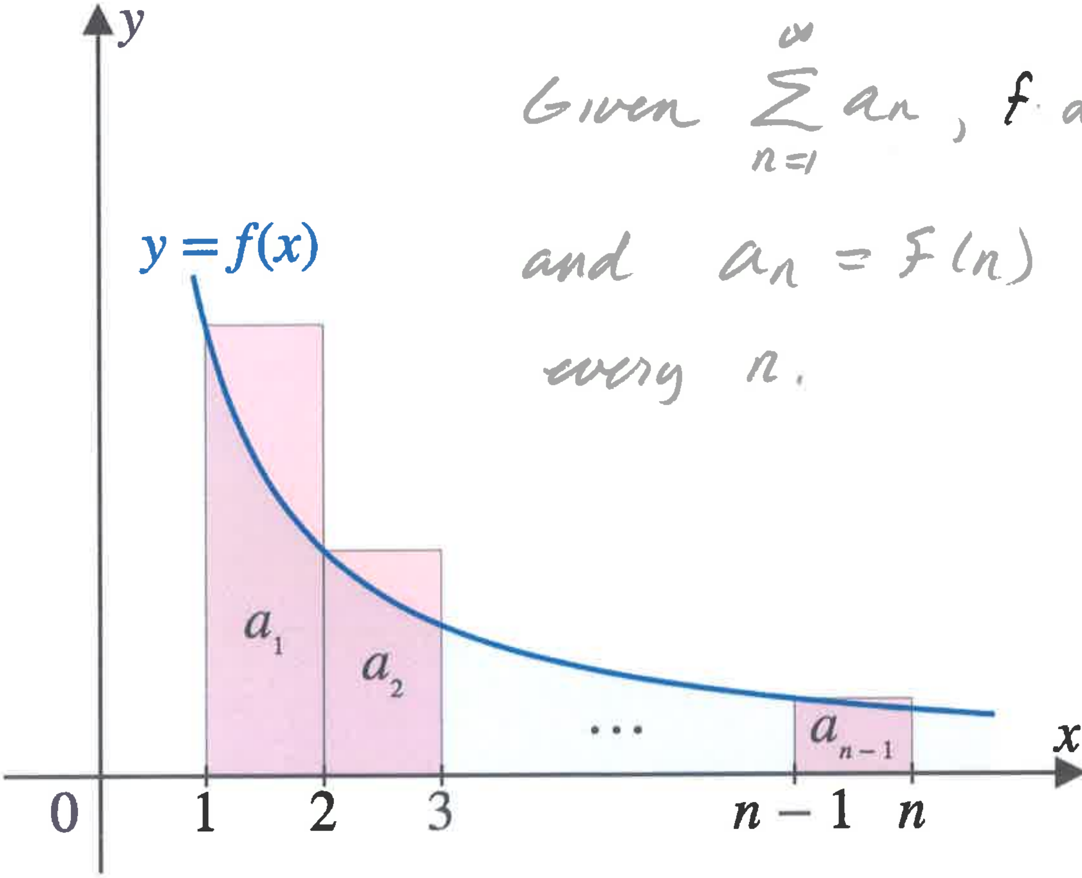
The Integral Test:

Given $\sum_{n=1}^{\infty} a_n$, f decreasing and $a_n = f(n)$ for each n.



If $\int_1^{\infty} f(x)\,dx < \infty$ then

$\sum_{n=1}^{\infty} a_n$ converges.



Given $\sum_{n=1}^{\infty} a_n$, f decreasing and $a_n = f(n)$ for every n.

If $\int_1^{\infty} f(x)\,dx = \infty$ then

$\sum_{n=1}^{\infty} a_n$ diverges.

Theorem: The Integral Test

Assume that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive terms and $a_n = f(n)$, where $f(x)$ is a continuous, positive, and decreasing function for all x greater than or equal to some natural number N. Then the series $\sum_{n=N}^{\infty} a_n$ and the improper integral $\int_N^{\infty} f(x)\ dx$ both converge or both diverge.

Example 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Let $f(x) = \frac{1}{x^2}$. Now $f(n) = \frac{1}{n^2}$ for each n and $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{n \to \infty} \int_1^n \frac{1}{x^2}\,dx$

$= \lim_{n \to \infty} \left(-\frac{1}{x}\right)\Big|_1^n = \lim_{n \to \infty} \left(-\frac{1}{n} + 1\right) = 1$

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example 2. Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

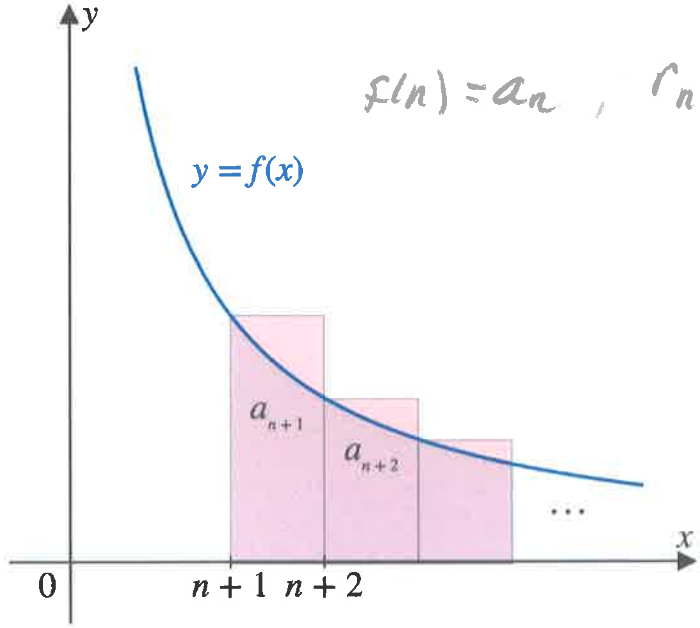
Let $f(x) = \frac{1}{x}$. Now $f(n) = \frac{1}{n}$ for each n

and $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{n \to \infty} \int_1^n \frac{1}{x}\,dx$

$= \lim_{n \to \infty} (\ln x)\Big|_1^n = \lim_{n \to \infty} \ln(n) = \infty$

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

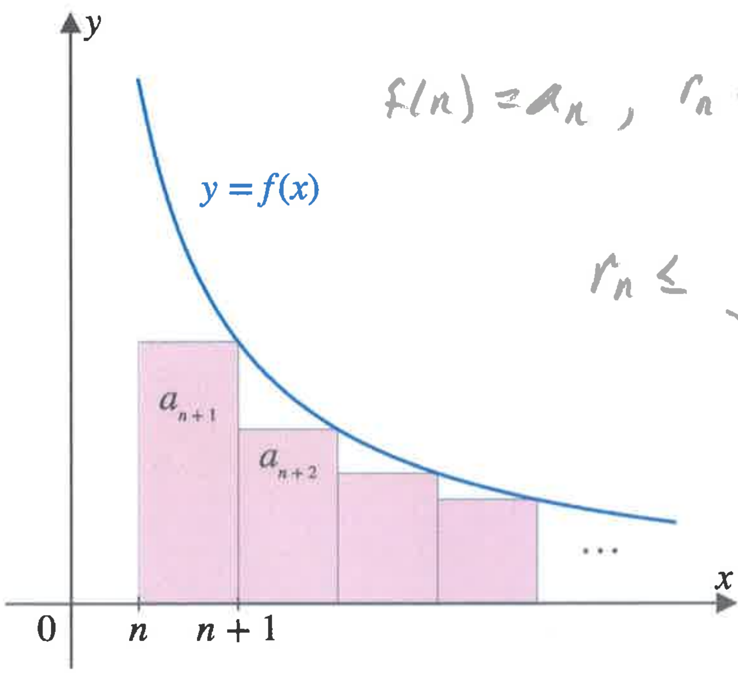
Series and Remainder Estimates:



$f(n) = a_n$, $r_n = \sum_{i=n+1}^{\infty} a_i$

$r_n \geq \int_{n+1}^{\infty} f(x)\,dx$

$$r_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx,$$



$f(n) = a_n$, $r_n = \sum_{i=n+1}^{\infty} a_i$

$r_n \leq \int_{n}^{\infty} f(x)\,dx$

$$s = \sum_{n=1}^{\infty} a_n = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i = s_n + r_n$$

$$r_n = a_{n+1} + a_{n+2} + \cdots \leq \int_{n}^{\infty} f(x)\,dx$$

Theorem: Series and Remainder Estimates

Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series and $a_n = f(n)$, where $f(x)$ is a continuous, positive, and decreasing function. Then the sum s, the n th partial sum s_n, and the n th remainder r_n satisfy the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \le r_n \le \int_{n}^{\infty} f(x)\,dx \qquad s_n + r_n = s$$

and

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx.$$

Error bounds on s_n

Example 3.

Use the Series and Remainder Estimates Theorem with five terms to find an interval containing s, the sum of the series $\sum_{n=1}^{\infty} \frac{19}{n^5}$. Round any intermediate calculations, if needed, to no less than six decimal places, and round your answers to five decimal places.

$n = 5, \quad S_5 = \sum_{n=1}^{5} \frac{19}{n^5}, \quad f(x) = \frac{19}{x^5}$

$S_5 = 27.906250, \quad \int_5^{\infty} \frac{19}{x^5} = .0076,$

$\int_6^{\infty} \frac{19}{x^5}\,dx = .0036651$

$27.90625 + .0036651 \le S \le 27.90625 + .0076$

$27.90992 \le S \le 27.91385$